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The Second Moment for the Meyer–König and Zeller Operators

J. A. H. ALKEMADE

Department of Mathematics, University of Technology, Delft, The Netherlands

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1. INTRODUCTION

In 1960 Meyer-König and Zeller [8] introduced a sequence of linear positive operators which they called Bernstein power series. Cheney and Sharma [2] modified these operators a little and these modified operators are now usually called, as we do here, the Meyer-König and Zeller operators.

Let $D[0, 1)$ be the set of real functions defined on $[0, 1)$ for which $|f(t)| \leqslant A(1-t)^{-\alpha}$ $(t \in [0, 1))$, where $A \geqslant 0$ and $\alpha \geqslant 0$ are constants which may both depend on f. Then the Meyer-König and Zeller operators M_n are defined on $D[0, 1)$ by

$$(M_n f)(x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k f\left(\frac{k}{n+k}\right) \qquad (x \in [0, 1); n \in \mathbb{N}). \tag{1.1}$$

If $f(1)$ exists and f is continuous to the left at 1, then (cf. [8])

$$\lim_{x \uparrow 1} (M_n f)(x) = f(1) \qquad (n \in \mathbb{N})$$

and $(M_n f)(1)$ is consequently defined by

$$(M_n f)(1) = f(1) \qquad (n \in \mathbb{N}). \tag{1.2}$$

If the functions e_i $(i = 0, 1, 2)$ are defined by $e_i : x \to x^i$ it is well known [2] that

$$(M_n e_0)(x) = 1 \qquad (x \in [0, 1]; n \in \mathbb{N})$$

and

$$(M_n e_1)(x) = x \qquad (x \in [0, 1]; n \in \mathbb{N}). \tag{1.3}$$

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In the case of the Bernstein, Szasz–Mirakjan and Baskakov operators it is easy to determine the image of e_2. For the Meyer-König and Zeller operators, however, an explicit expression for $(M_n e_2)(x)$ does not yet occur in the literature. Many authors are only dealing with *estimations* of the second moment $(M_n e_2)(x) - x^2$: Müller [9], Sikkema [12], Lupaş and Müller [6] and Becker and Nessel [1] to mention some of them in chronological order.

In Section 3 an *explicit* expression for $(M_n e_2)(x)$ is determined in terms of a convergent power series in x, which in particular is a hypergeometric series. The way of deriving this expression is based upon a differential equation which is satisfied by the right-hand side of (1.1). This differential equation is determined in Section 2. In the last two sections several applications are given. In Section 4 it is shown that some of the known estimates for $(M_n e_2)(x) - x^2$ immediately follow from the explicit expression (3.3) for $(M_n e_2)(x)$. In Section 5 an estimate of the sup-norm of $M_n e_2 - e_2$ on $[0, 1]$, which was given by Sikkema [12], will be improved. This improvement is twofold: the new estimate is better and of a more handy form. It will be used to improve upon several known theorems on the Meyer-König and Zeller operators.

Some of the results of this paper were the subject of the author's lecture at the Second Edmonton Conference on Approximation Theory, June 1982.

2. A Differential Equation

In this section a differential equation is derived which will be the starting point for the determination of $(M_n e_2)(x)$ in Section 3. Differential equations like the one given in Theorem 1 for the Meyer-König and Zeller operators, seem to be fundamental for the investigation of many kinds of linear positive operators. Some papers, in which equations analogous to the one in Theorem 1 are given, are Walk [15], Götz [3], May [7], Ismail and May [4], Ismail [5] and Volkov [14]. In these papers the main use of such an equation confines itself to a classification and a simultaneous treatment of different operators. Special properties of one specific operator are not taken into account, as is done as a matter of fact in the present paper.

THEOREM 1. *Let*

$$g(t) = \frac{t}{1-t} \qquad (t \in [0, 1)). \tag{2.1}$$

For each $n \in \mathbb{N}$, $x \in [0, 1)$ *and* $f \in D[0, 1)$, $(M_n f)(x)$ *as defined in* (1.1) *satisfies the differential equation*

$$x(1-x)\frac{d}{dx}(M_n f)(x)$$
$$= -(n+1)\,x(M_n f)(x) + n(1-x)(M_n(gf))(x). \tag{2.2}$$

Remark. Strictly speaking (2.2) is not a differential equation for $(M_n f)(x)$ but rather a functional-differential equation.

Proof. Let $n \in \mathbb{N}$. Because $f \in D[0, 1)$ the power series on the right-hand side of (1.1) converges on $[0, 1)$. Hence it is allowed to differentiate this series term by term in $[0, 1)$. Thus

$$\frac{d}{dx}(M_n f)(x) = -(n+1)(1-x)^n \sum_{k=0}^{\infty} \binom{n+k}{k} x^k f\left(\frac{k}{n+k}\right)$$
$$+ (1-x)^{n+1} \sum_{k=1}^{\infty} \binom{n+k}{k} k x^{k-1} f\left(\frac{k}{n+k}\right).$$

Multiplying this equation by $x(1-x)$ and using $g(k/(k+n)) = k/n$, which is apparent from (2.1), it follows that

$$x(1-x)\frac{d}{dx}(M_n f)(x)$$
$$= -(n+1)\,x(1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k f\left(\frac{k}{n+k}\right)$$
$$+ n(1-x)^{n+2} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k g\left(\frac{k}{n+k}\right) f\left(\frac{k}{n+k}\right).$$

By (1.1) the theorem is proved.

Theorem 1 implies

LEMMA 1. *For each $n \in \mathbb{N}$, $M_n e_2$ is a solution of the differential equation*

$$x(1-x)\,y'(x) + (n+x)\,y(x) = nx^2 + x \qquad (x \in [0, 1)), \tag{2.3}$$

which satisfies the condition $y(0) = 0$.

Proof. Let $n \in \mathbb{N}$ and $x \in [0, 1)$. It is clear from (1.1) that $(M_n e_2)(0) = 0$. Setting in (2.2) $f = e_1 - e_2$ it follows that

$$x(1-x)\frac{d}{dx}(M_n(e_1-e_2))(x) = -(n+1)\,x(M_n(e_1-e_2))(x)$$
$$+ n(1-x)(M_n e_2)(x).$$

Using the linearity of M_n and (1.3) it is seen that $M_n e_2$ indeed satisfies (2.3). Thus the lemma is proved.

3. An Explicit Expression for $(M_n e_2)(x)$

It is pointed out in the Introduction that an explicit expression for $(M_n e_2)(x)$ will be derived in this section. This expression follows from Lemma 1 and is given in Theorem 2. The following notation will be used:

$$(y)_0 = 1,$$

$$(y)_k = \prod_{i=0}^{k-1} (y+i) \qquad (k \in \mathbb{N}, y \in \mathbb{R}).$$

Furthermore some properties of the hypergeometric series

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where a, b, $c \in \mathbb{R}$ and $c \neq 0, -1, -2, \ldots$, are needed. This series is convergent for $|x| < 1$ and if $c - a - b > 0$ also for $x = 1$. The function represented by the sum of the convergent series is denoted by ${}_2F_1(a, b; c; x)$. Thus

$$ {}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \qquad (|x| < 1) \tag{3.1}$$

and

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad (c - a - b > 0). \tag{3.2}$$

(cf. [16, pp. 281–282]).

Theorem 2. *For $n \in \mathbb{N}$ the formula*

$$(M_n e_2)(x) = x^2 + \frac{x(1-x)^2}{n+1}\, {}_2F_1(1, 2; n+2; x) \tag{3.3}$$

holds for each $x \in [0, 1)$. For $n \geqslant 2$, (3.3) also holds at $x = 1$.

Proof. Let $n \in \mathbb{N}$ and $x \in [0, 1)$. By the substitution of

$$y(x) = x^2 + x(1-x)^2 z(x)$$

into (2.3) it follows that z satisfies the differential equation

$$x(1-x)\, z'(x) + (n + 1 - 2x)\, z(x) = 1 \ (x \in [0, 1)). \tag{3.4}$$

A particular solution z_p of this equation can be found by substituting

$$z(x)=\sum_{k=0}^{\infty} a_k x^k$$

into (3.4) and comparing powers of x. This leads to

$$z_p(x)=\frac{1}{n+1}\sum_{k=0}^{\infty}\frac{(2)_k}{(n+2)_k}x^k=\frac{1}{n+1}\,{}_2F_1(1,2;n+2;x).$$

Furthermore, let z_h denote the general solution of the homogeneous equation of (3.4) valid on (0, 1). Then

$$z_h(x)=Cx^{-n-1}(1-x)^{n-1}\ (C\in\mathbb{R}).$$

The general solution of (2.3) valid on (0, 1) is therefore

$$y(x)=x^2+\frac{x(1-x)^2}{n+1}\,{}_2F_1(1,2;n+2;x)+Cx^{-n}(1-x)^{n+1}\ (C\in\mathbb{R}).$$

As $M_n e_2$ ($n\in\mathbb{N}$) equals one of these solutions and $(M_n e_2)(0)=0$ it follows that $C=0$ which gives (3.3). Using (3.2) it follows that (3.3) also holds at $x=1$ if $n\geqslant 2$.

The next lemma will be used several times in Sections 4 and 5.

LEMMA 2. *For $n\geqslant 2$ and $x\in[0,1]$ there holds for any $m\in\mathbb{N}$,*

$${}_2F_1(1,2;n+2;x)\leqslant\sum_{k=0}^{m-1}\frac{(2)_k}{(n+2)_k}x^k+\frac{(m+1)!\,x^m}{(n-1)(n+2)_{m-1}}.$$

Proof. In view of (3.1) the proof of this lemma will consist of a proper estimation of

$$\phi_m(x)=\sum_{k=m}^{\infty}\frac{(2)_k}{(n+2)_k}x^k\qquad(n\geqslant 2, x\in[0,1)).$$

$\phi_m(x)$ can be written as

$$\begin{aligned}\phi_m(x)&=\frac{(2)_m}{(n+2)_m}x^m\sum_{k=m}^{\infty}\frac{(m+2)_{k-m}}{(n+2+m)_{k-m}}x^{k-m}\\&=\frac{(m+1)!}{(n+2)_m}x^m\sum_{k=0}^{\infty}\frac{(m+2)_k}{(n+m+2)_k}x^k\\&=\frac{(m+1)!}{(n+2)_m}x^m\,{}_2F_1(1,m+2;n+m+2;x).\end{aligned}$$

Hence

$$\phi_m(x) \leqslant \frac{(m+1)!}{(n+2)_m} x^m \,{}_2F_1(1, m+2; n+m+2; 1),$$

because $n \geqslant 2$. Equation (3.2) then gives

$$\phi_m(x) \leqslant \frac{(m+1)!}{(n+2)_m} x^m \frac{n+m+1}{n-1} = \frac{(m+1)!\, x^m}{(n-1)(n+2)_{m-1}}.$$

Thus Lemma 2 is established.

4. Some Known Estimates of the Second Moment

In this section the expression (3.3) for $(M_n e_2)(x)$ in terms of a hypergeometric series is used to derive some known estimates for the second moment $(M_n e_2)(x) - x^2$. First, ${}_2F_1(1, 2; n+2; x)$ can easily be estimated by a geometric series if $x \in [0, 1)$. Indeed, by (3.1) for each $n \in \mathbb{N}$

$$ {}_2F_1(1, 2; n+2; x) = \sum_{k=0}^{\infty} \frac{(2)_k}{(n+2)_k} x^k \tag{4.1}$$

and thus

$$ {}_2F_1(1, 2; n+2; x) \leqslant \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

if $x \in [0, 1)$ and Theorem 2 then leads to the estimate

$$0 \leqslant (M_n e_2)(x) - x^2 \leqslant \frac{x(1-x)}{n+1} \qquad (x \in [0, 1)). \tag{4.2}$$

Because $(M_n e_2)(1) = 1$ by (1.2), (4.2) also holds at $x = 1$. In 1967 Müller [9, p. 61] also proved this result.

Taking into account only the first two terms of the series in (4.1), a lower bound for the second moment is obtained. Together with an application of Lemma 2 with $m = 1$, (3.3) leads to

$$\frac{x(1-x)^2}{n+1}\left(1 + \frac{2x}{n+2}\right) \leqslant (M_n e_2)(x) - x^2 \leqslant \frac{x(1-x)^2}{n+1}\left(1 + \frac{2x}{n-1}\right), \tag{4.3}$$

if $n \geqslant 2$ and $x \in [0, 1]$. These inequalities were also derived by Becker and Nessel [1] in 1978.

Applying Lemma 2 with $m=2$ and using the same lower bound as in (4.3) it follows that

$$(M_n e_2)(x) - x^2 = \frac{x(1-x)^2}{n+1} + \frac{2x^2(1-x)^2}{(n+1)(n+2)} + R_n(x), \tag{4.4}$$

where

$$0 \leqslant R_n(x) \leqslant \frac{6x^3(1-x)^2}{(n-1)(n+1)(n+2)}. \tag{4.5}$$

Therefore

$$(M_n e_2)(x) - x^2 = \frac{x(1-x)^2}{n} + \frac{x(1-x)^2\,(2x-1)}{n^2} + S_n(x), \tag{4.6}$$

where

$$S_n(x) = \mathcal{O}\left(\frac{1}{n^3}\right) \qquad (n \to \infty). \tag{4.7}$$

By a closer analysis it can be seen from (4.4) and (4.5) that even

$$|S_n(x)| \leqslant \frac{5x(1-x)^2}{n^3} < \frac{1}{n^3} \qquad (n \geqslant 2). \tag{4.8}$$

Expression (4.6) together with the asymptotic estimation (4.7) for $S_n(x)$ can also be found in a paper by Sikkema [12] of 1970. Estimation (4.8) for $S_n(x)$, however, seems to be new.

5. Improvement of Some Theorems concerning the M_n-Operators

In this section a lemma, proved by Sikkema [12] (Lemma 3 below), will be improved. This improvement stated in Theorem 3 will in turn lead to an improvement of several theorems on the Meyer–König and Zeller operators.

In what follows the second moment of the M_n-operators is denoted by $F_n(x)$, thus

$$F_n(x) = (M_n e_2)(x) - x^2 \qquad (x \in [0, 1],\ n \in \mathbb{N}). \tag{5.1}$$

Let further $\|f\|$ denote the supremum norm of $f \in C\,[0, 1]$.

LEMMA 3 (Sikkema). *Let F_n be defined by* (5.1). *Then*

(a) $\|F_1\| \leqslant 0{,}1113$,

(b) $\|F_n\| \leqslant (4/27n)(1-((n^2-5)/4(n^2-1)^2)) \qquad (n \geqslant 2)$.

THEOREM 3. *For F_n defined in* (5.1) *there holds*

(a) $\|F_1\| = 0.0999032$ (*exact up to the last digit shown*),

(b) $\|F_n\| \leqslant 4/(27n+9) \qquad (n \geqslant 2)$,

(c) $\|F_n\| = (4/27n) - (4/81n^2) + \mathcal{O}(n^{-3}) \qquad (n \to \infty)$.

Proof. Because $F_n(0) = F_n(1) = 0$ and $F_n(x) > 0$ on $(0,1)$ (cf (4.3)) for each $n \in \mathbb{N}$, the maximal value of $F_n(x)$ is attained at some point $x_0 \in (0, 1)$. Substituting $y(x) = x^2 + F_n(x)$ into (2.3), $F_n(x)$ is seen to satisfy

$$x(1-x)\,F_n'(x) + (n+x)\,F_n(x) = x(1-x)^2. \tag{5.2}$$

As $F_n'(x_0) = 0$ it follows that

$$(n+x_0)\,F_n(x_0) = x_0(1-x_0)^2,$$

which in view of Theorem 2 and the fact that $x_0 \in (0, 1)$ is equivalent to

$${}_2F_1(1, 2; n+2; x_0) = \frac{n+1}{n+x_0}. \tag{5.3}$$

Let $f_n(x)$ $(x \in [0, 1],\ n \in \mathbb{N})$ be defined as

$$f_n(x) = \frac{n+1}{n+x}. \tag{5.4}$$

Now, ${}_2F_1(1, 2; n+2; x)$ is monotonically increasing on $[0, 1)$ from 1 at $x = 0$ to ∞ if $n = 1$ and $x \to 1$ and to $(n+1)/(n-1)$ if $x \to 1$ and $n \geqslant 2$. Furthermore, for each $n \in \mathbb{N} f_n(x)$ is monotonically decreasing on $[0, 1]$ from $(n+1)/n$ at $x = 0$ to 1 at $x = 1$. Consequently there exists only one value $x_0 \in (0, 1)$ for which (5.3) holds. Concerning n two cases will be distinguished. First, let $n = 1$. Then by (5.3) x_0 satisfies the equation

$${}_2F_1(1, 2; 3; x) = \frac{2}{x+1}. \tag{5.5}$$

From (4.1) it follows that if $x \in (0, 1)$

$$\begin{aligned} {}_2F_1(1, 2; 3; x) &= \sum_{k=0}^{\infty} \frac{2}{k+2} x^k = 2x^{-2} \sum_{k=0}^{\infty} \frac{x^{k+2}}{k+2} \\ &= 2x^{-2} \left\{ \sum_{k=1}^{\infty} \frac{x^k}{k} - x \right\} = -2x^{-2}\{\ln(1-x) + x\}. \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) it follows that x_0 is the solution of the equation

$$-\frac{1}{x}\ln(1-x)-1=\frac{x}{1+x}$$

on (0, 1). Using a table for $\ln x$ it can be deduced from this equality that $x_1 < x_0 < x_2$ with $x_1 = 4176.10^{-4}$ and $x_2 = 4177.10^{-4}$. Equation (5.2) yields

$$F_1'(x) = (1-x)\{1-\tfrac{1}{2}(x+1)\,{}_2F_1(1,2;3;x)\}, \tag{5.7}$$

which shows that F_1' is monotonically decreasing on $[0, x_0]$. Moreover, $F_1'(x) > 0$ for $x_1 \leqslant x < x_0$ which leads to

$$F_1(x_1) < F_1(x_0) < F_1(x_1) + F_1'(x_1)(x_2 - x_1).$$

Substituting the values for x_1 and x_2 in this formula, using (5.1), (3.3), (5.6) and (5.7) it follows that

$$999032.10^{-7} < F_1(x_0) < 9990325.10^{-8}.$$

Thus part (a) of Theorem 3 has been proved.

Second, let $n \geqslant 2$. Application of Lemma 2 with $m = 2$ gives

$${}_2F_1(1,2;n+2;x) \leqslant 1 + \frac{2x}{n+2} + \frac{6x^2}{(n-1)(n+2)}.$$

Hence if $n \geqslant 3$ the following estimation holds

$$\begin{aligned} {}_2F_1(1,2;n+2;\tfrac{1}{3}) &\leqslant 1 + \frac{(2/3)n}{n^2+n-2} \\ &\leqslant 1 + \frac{(2/3)\,n}{n^2+\frac{1}{3}n} = 1 + \frac{2/3}{n+\frac{1}{3}} = f_n\left(\frac{1}{3}\right), \end{aligned} \tag{5.8}$$

by (5.4). Because ${}_2F_1(1,2;n+2;x)$ is monotonically increasing and $f_n(x)$ is monotonically decreasing on [0, 1) it follows from (5.8) that

$${}_2F_1(1,2;n+2;x_0) \leqslant f_n\left(\frac{1}{3}\right) = \frac{n+1}{n+1/3} \qquad (n \geqslant 3).$$

If $n \geqslant 3$ the latter estimation leads with (5.1) and (3.3) to

$$\begin{aligned} \|F_n\| = F_n(x_0) &= \frac{1}{n+1}\, x_0(1-x_0)^2\, {}_2F_1(1,2;n+2;x_0) \\ &\leqslant \frac{1}{n+1} \max_{x\in[0,1]} (x(1-x)^2)\, \frac{n+1}{n+1/3} = \frac{4}{27n+9}, \end{aligned} \tag{5.9}$$

which is part (b) of Theorem 3. If $n=2$ the second inequality in (5.8) does not hold. However, application of Lemma 2 with $m=3$ and $n=2$ yields

$$ {}_2F_1(1, 2; 4; \tfrac{1}{3}) \leqslant 1 + \frac{2/3}{4} + \frac{2/3}{20} + \frac{24/27}{20} < \frac{5}{4}, $$

and because $f_2(\frac{1}{3}) = \frac{9}{7} > \frac{5}{4}$, (5.9) also holds if $n=2$.

It remains to prove part (c) of Theorem 3. Solving Eq. (5.3) asymptotically for $n \to \infty$ it follows that

$$ x_0 = \frac{1}{3} + \frac{4}{27n} + \mathcal{O}(n^{-2}) \qquad (n \to \infty). $$

Substitution of this expression for x_0 into (3.3) and use of (4.1) give

$$ \|F_n\| = \frac{4}{27n} - \frac{4}{81n^2} + \mathcal{O}(n^{-3}) \qquad (n \to \infty), $$

which is part (c) of the theorem.

Remark. The asymptotic expansion for $n \to \infty$ of the upper bound of $\|F_n\|$ given in part (b) of Theorem 3 coincides up to the order $\mathcal{O}(n^{-3})$ with the asymptotic expression given in part (c). This means that if $n \to \infty$ the inequality in part (b) of Theorem 3 is an equality up to $\mathcal{O}(n^{-3})$.

It is obvious that part (a) of Theorem 3 indeed gives an improvement of the corresponding part of Lemma 3. Also part (b) of Theorem 3 improves upon the corresponding part of Lemma 3. To show this it is enough to prove the inequality

$$ \frac{1}{n} - \frac{n^2-5}{4n(n^2-1)^2} > \frac{1}{n+1/3} \qquad (n \geqslant 2). \tag{5.10} $$

In fact, if $n \geqslant 2$ then

$$ \frac{1}{n} - \frac{n^2-5}{4n(n^2-1)^2} > \frac{1}{n} - \frac{1}{4n(n^2-1)} \geqslant \frac{1}{n} - \frac{1}{4n(2n-1)} $$
$$ > \frac{1}{n} - \frac{1}{4n((3/4)n+(1/4))} = \frac{1}{n+1/3}. $$

Theorems 2 and 3 will now be used to improve upon some theorems on Meyer-König and Zeller operators occurring in the literature. First, a special case of a theorem of Shisha and Mond [11] is stated:

THEOREM 4. (Shisha and Mond). *Let* $L_n: C[0,1] \to C[0,1]$ $(n \in \mathbb{N})$ *be* *a sequence of linear positive operators satisfying* $L_n e_i = e_i$ $(i = 0, 1)$. *Then for any* $\delta > 0$

$$|(L_n f)(x) - f(x)| \leqslant \{1 + \delta^{-2}((L_n e_2)(x) - x^2)\}\, \omega(f; \delta), \qquad (5.11)$$

where $\omega(f; \delta)$ *denotes the modulus of continuity of* f *on* $[0, 1]$.

As a consequence of Theorems 2 and 4, with $\delta = (n+1)^{-1/2}$ in (5.11), the pointwise estimate in the next theorem holds.

THEOREM 5. *Let* $f \in C[0, 1]$. *Then for the operators* M_n *defined by* (1.1) *the inequality*

$$|(M_n f)(x) - f(x)| \leqslant \{1 + x(1-x)^2\, {}_2F_1(1, 2; n+2; x)\}\, \omega\left(f; \frac{1}{\sqrt{n+1}}\right)$$

holds if $x \in [0, 1]$ *and* $n \geqslant 2$.

Application of Theorem 4 to the Meyer-König and Zeller operators, with $\delta = n^{-1/2}$ in (5.11) and use of Theorem 3(b) lead to

THEOREM 6. *Let* $f \in C[0, 1]$, *then for* $n \geqslant 2$

$$\|M_n f - f\| \leqslant \left\{1 + \frac{4n}{27n + 9}\right\} \omega\left(f; \frac{1}{\sqrt{n}}\right).$$

Because of (5.10) this is a sharper result than that in Theorem 6 in [12], which in turn is an improvement of the estimate

$$\|M_n f - f\| \leqslant \frac{31}{27}\, \omega\left(f; \frac{1}{\sqrt{n}}\right),$$

occurring in [6]. The result in Theorem 6 is better than the older ones given in [6, 12] because use was made of the sharper estimate for $\|M_n e_2 - e_2\|$ mentioned in Theorem 3. In a similar way several theorems on Meyer-König and Zeller operators can be improved, which will be indicated now by two other applications of part (b) of Theorem 3. For the first one the next theorem occurring in [6] is needed.

THEOREM 7 (Lupaş and Müller). *Let* $L_n : C[0, 1] \to C[0, 1]$ $(n \in \mathbb{N})$ *be a sequence of linear positive operators satisfying* $L_n e_i = e_i$ $(i = 0, 1)$. *If* f' *exists and is continuous on* $[0, 1]$, *then for any* $\delta > 0$

$$\|L_n f - f\| \leqslant (1 + \delta^{-1}) \|L_n e_2 - e_2\|\, \omega(f'; \delta).$$

This theorem applied with $\delta = n^{-1/2}$ to the M_n-operators together with Theorem 3 yields the Lorentz-type theorem,

THEOREM 8. *If f is such that $f' \in C[0, 1]$ then*

$$\|M_n f - f\| \leqslant \left\{1 + \frac{2}{3}\sqrt{\frac{n}{3n+1}}\right\} \frac{2}{3\sqrt{3n+1}}\,\omega\left(f'; \frac{1}{\sqrt{n}}\right) \quad (n \geqslant 2).$$

This estimate is clearly a better one than

$$\|M_n f - f\| \leqslant \left\{1 + \frac{2}{3\sqrt{3}}\right\} \frac{2}{3\sqrt{3n}}\,\omega\left(f'; \frac{1}{\sqrt{n}}\right),$$

which has been proved by Lupaş and Müller [6].

The last application of Theorem 3 leads to an improvement of a theorem proved by Singh [13]. It gives an estimate of $\|(M_n f)' - f'\|$ in terms of the modulus of continuity of f'' and was already an improvement of a theorem of Müller [10]. In the proof of this theorem Singh uses the estimation of Sikkema, mentioned in Lemma 3. On replacing in Singh's proof this estimation by that of parts (a) and (b) of Theorem 3 the following theorem can be proved.

THEOREM 9. *Let f be such that $f'' \in C\,[0, 1]$, then*

$$\|(M_n f)' - f'\| \leqslant d_n \omega\left(f''; \frac{1}{\sqrt{n-1}}\right) + \frac{\|f''\|}{n} \quad (n \geqslant 2),$$

where

$d_2 = 1,149$ *(exact up to the last digit shown),*

$$d_n = \frac{2}{3\sqrt{3n-2}}\left\{1 + \frac{1}{3}\sqrt{\frac{n-1}{3n-2}} + \frac{\sqrt{n-1}}{n}\right\} + \frac{1}{n} + \frac{\sqrt{n-1}}{2n^2} \quad (n \geqslant 3).$$

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